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Discrete Mathematics 306 (2006) 2327–2335

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# On an adjacency property of almost all tournaments

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Received 3 March 2003; received in revised form 15 September 2003; accepted 15 September 2003

Available online 27 July 2006

In loving memory of Claude Berge

## Abstract

Let  $n$  be a positive integer. A tournament is called  $n$ -existentially closed (or  $n$ -e.c.) if for every subset  $S$  of  $n$  vertices and for every subset  $T$  of  $S$ , there is a vertex  $x \notin S$  which is directed toward every vertex in  $T$  and directed away from every vertex in  $S \setminus T$ . We prove that there is a 2-e.c. tournament with  $k$  vertices if and only if  $k \geq 7$  and  $k \neq 8$ , and give explicit examples for all such orders  $k$ . We also give a replication operation which preserves the 2-e.c. property.

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MSC: 05C35; 05C80; 05C75

Keywords: Adjacency property;  $n$ -Existentially closed; Tournament; Replication

## 1. Introduction

A *tournament* is a directed graph with exactly one arc between each pair of distinct vertices. Consider the following *adjacency property* for tournaments.

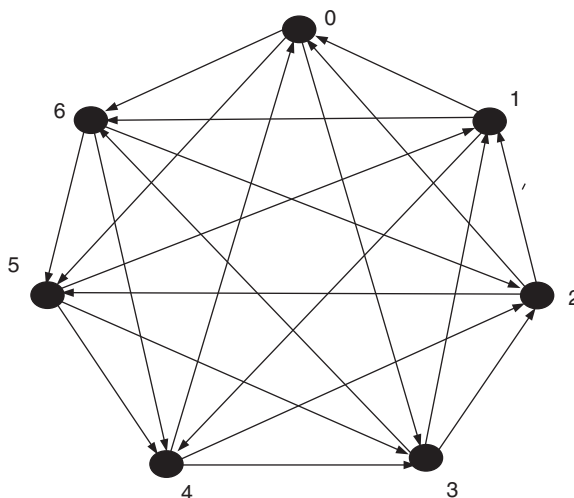
**Definition 1.** Let  $n$  be a positive integer. A tournament is called  $n$ -existentially closed or  $n$ -e.c. if for every  $n$ -element subset  $S$  of the vertices, and for every subset  $T$  of  $S$ , there is a vertex  $x \notin S$  which is directed toward every vertex in  $T$  and directed away from every vertex in  $S \setminus T$ . (Note that  $T$  may be empty.)

Adjacency properties of tournaments were studied in [3,8,15,18,23]. Much of the research on such properties is motivated by the fact that while almost all tournaments (with arcs chosen independently and with probability  $p$ , where  $0 < p < 1$  is a fixed real number) are  $n$ -e.c. for any fixed positive integer  $n$  (see [15]), few *explicit* examples of such tournaments are known.

Adjacency properties of graphs were studied by numerous authors; see [9] for a survey. A graph is called  $n$ -existentially closed or  $n$ -e.c. if it satisfies the following adjacency property: for every  $n$ -element subset  $S$  of the vertices, and for every subset  $T$  of  $S$ , there is a vertex not in  $S$  which is joined to every vertex of  $T$  and to no vertex of  $S \setminus T$ . The  $n$ -e.c. property is of

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Fig. 1. The tournament  $D_7$ .

interest in part because the countable random graph is  $n$ -e.c. for all  $n \geq 1$ ; in fact, the countable random graph is the unique (up to isomorphism) countable graph that is  $n$ -e.c. for all  $n \geq 1$ . The countable random tournament is the analogue of the random graph for tournaments; see [13]. The countable random tournament is the unique (up to isomorphism) countable tournament that is  $n$ -e.c. for all  $n \geq 1$ .

The cases  $n = 1, 2$  for graphs were studied in [9,10,12]. For  $n > 2$ , few explicit examples of  $n$ -e.c. graphs are known other than large Paley graphs (see [2,8]). A prolific construction of  $n$ -e.c. graphs for all  $n$  was recently given in [14].

In the present article, we concentrate on the 2-e.c. adjacency property. Note that a tournament is 2-e.c. if the following adjacencies hold: for every pair of vertices,  $u$  and  $v$ , there are four other vertices: one directed toward both  $u$  and  $v$ , one directed away from both  $u$  and  $v$ , one directed toward  $u$  and away from  $v$ , and one directed toward  $v$  and away from  $u$ . In Section 3, we prove that there is a 2-e.c. tournament with  $k$  vertices if and only if  $k \geq 7$  and  $k \neq 8$ , and give explicit examples for all such orders  $k$ .

We consider only finite and simple tournaments. For a tournament  $G$ ,  $V(G)$  denotes its vertex-set and  $E(G)$  denotes its arc-set. The order of  $G$  is  $|V(G)|$ . We denote an arc directed from  $x$  to  $y$  by  $(x, y)$ . For a vertex  $x \in V(G)$ , we define  $N_{\text{out}}(x) = \{y : (x, y) \in E(G)\}$ , and  $N_{\text{in}}(x) = \{y : (y, x) \in E(G)\}$ . As usual, a vertex  $x$  with  $N_{\text{in}}(x) = \emptyset$  is called a source and a vertex  $x$  with  $N_{\text{out}}(x) = \emptyset$  is called a sink. If  $U \subseteq V(G)$ ,  $G|U$  is the subgraph of  $G$  induced by  $U$ ; for  $x \in V(G)$ ,  $G - x = G|(V(G) \setminus \{x\})$ . For basic information on graphs and tournaments, see [4,11].

The Paley tournament of order  $q$ , written  $D_q$ , where  $q$  is a prime power congruent to 3 (mod 4), is the tournament with vertices the elements of  $GF(q)$ , the finite field with  $q$  elements, and  $(x, y) \in E(D_q)$  if and only if  $x - y$  is a nonzero quadratic residue. For  $D_7$ , see Fig. 1. As discussed above for Paley graphs, for a fixed positive  $n$ , sufficiently large Paley tournaments are  $n$ -e.c. (see [18]); however, no other explicit families of tournaments with these adjacency properties are known.

The next lemma follows from the definitions.

**Lemma 1.** Let  $G$  be an  $n$ -e.c. tournament for some  $n > 1$ . For a fixed  $v \in V(G)$ , the tournaments  $G - v$ ,  $G|N_{\text{in}}(v)$ , and  $G|N_{\text{out}}(v)$  are each  $(n - 1)$ -e.c.

**Definition 2.** A tournament  $G$  is  $n$ -e.c. minimal if  $G$  has the smallest number of vertices among all  $n$ -e.c. tournaments. An  $n$ -e.c. tournament is critical if deleting any vertex leaves a tournament which is not  $n$ -e.c.

Clearly, an  $n$ -e.c. minimal tournament is  $n$ -e.c. critical. In Section 2, we show that there are exactly two 1-e.c. critical tournaments up to isomorphism. In Section 4, we give examples of 2-e.c. critical tournaments of all possible

orders  $k \geq 7$  and  $k \neq 8$ . Vertex-criticality for various properties has been studied by many authors, including Berge [6,5,7,1,17,20–22,24,25].

## 2. The 1-e.c. critical tournaments

We make the following trivial observations.

**Remark 1.** A tournament is 1-e.c. if and only if it has no source or sink.

**Remark 2.** A tournament with a directed hamilton cycle is 1-e.c.

The tournament  $D_3$  is the directed circuit on three vertices. It is easy to see that  $D_3$  is the unique (up to isomorphism) 1-e.c. minimal tournament, and thus, it is 1-e.c. critical. Define  $T_6$  to be the tournament consisting of two copies of  $D_3$ , with arcs oriented from the first copy to the second. It is straightforward to check that  $T_6$  is 1-e.c. critical.

**Theorem 2.** *The only 1-e.c. critical tournaments (up to isomorphism) are  $D_3$  and  $T_6$ .*

**Proof.** Let  $G$  be a 1-e.c. critical tournament. We first observe that a strongly connected component  $S$  of  $G$  has exactly three vertices. To see this, suppose that  $S$  has at least  $k \geq 4$  vertices. By a theorem of Moon [19],  $S$  has a directed circuit  $C$  of length  $k - 1$ . Deleting the vertex that  $C$  misses in  $S$  leaves a 1-e.c. tournament, which is a contradiction.

We claim that if  $G$  has exactly one or two strongly connected components, then  $G$  is isomorphic to  $D_3$  or  $T_6$ , respectively. Assume to the contrary that  $G$  has  $r \geq 3$  strongly connected components. From  $G$  we construct an auxiliary tournament  $G'$ , whose vertices are the strongly connected components of  $G$  with the induced adjacencies. Note that  $G'$  is isomorphic to the  $r$ -element linear order. Let  $u$  be a vertex of  $G'$  that is neither a least nor greatest element. If we delete a vertex  $x$  in the strongly connected component of  $G$  corresponding to  $u$ , then the remaining graph  $G - x$ , is 1-e.c., which is a contradiction.  $\square$

## 3. Examples of 2-e.c. tournaments

In this section, our main theorem is the following.

**Theorem 3.** *There is a 2-e.c. tournament with  $k$  vertices if and only if  $k \geq 7$  and  $k \neq 8$ .*

To prove Theorem 3, we first prove the following theorem.

**Theorem 4.** *There is a unique (up to isomorphism) 2-e.c. minimal tournament, the Paley tournament  $D_7$ .*

**Proof.** Let  $G$  be a 2-e.c. tournament. Then since the unique minimal 1-e.c. tournament has three vertices,  $|V(G)| \geq 7$  by Lemma 1. Suppose now  $|V(G)| = 7$ , say  $V(G) = \{1, 2, 3, 4, 5, 6, 7\}$ . Say  $N_{\text{in}}(7) = \{1, 2, 3\}$ ,  $(1, 2), (2, 3), (3, 1) \in E(G)$ ;  $N_{\text{out}}(7) = \{4, 5, 6\}$ ;  $(4, 6), (6, 5), (5, 4) \in E(G)$ . See Fig. 2(a). Vertex 1 currently has outdegree two, but needs outdegree three, so without loss of generality, assume that  $(1, 4) \in E(G)$ . Then by considering the degrees of 1 and 4, we get  $(5, 1), (6, 1) \in E(G)$  and  $(4, 2), (4, 3) \in E(G)$ . See Fig. 2(b). Since  $N_{\text{in}}(1) = \{3, 5, 6\}$  and  $(6, 5) \in E(G)$ , it follows that  $(5, 3), (3, 6) \in E(G)$ . See Fig. 2(c). Then, for degree of 5,  $(2, 5) \in E(G)$ , and then for degree of 2,  $(6, 2) \in E(G)$ . See Fig. 2(d). Then  $f : V(G) \rightarrow V(D_7)$  is an isomorphism, where  $f(1) = 0, f(2) = 5, f(3) = 4, f(4) = 6, f(5) = 1, f(6) = 2$  and  $f(7) = 3$ .  $\square$

Given a 2-e.c. tournament, another 2-e.c. tournament with two more vertices can be constructed using a “tournament version” of the replication operation which was instrumental in [9].

**Definition 3.** Let  $G$  be a tournament and let  $(a, b) \in E(G)$ . Add two new vertices  $a', b'$  such that  $a'$  has the same adjacencies to vertices of  $G$  other than  $b$  as  $a$  does,  $b'$  has the same adjacencies to vertices of  $G$  other than  $a$  as  $b$  does,

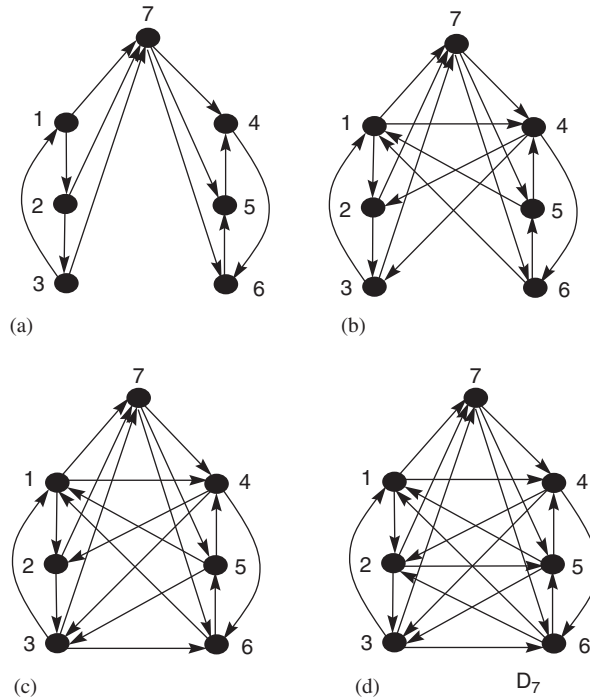


Fig. 2. The proof of Theorem 4.

$a, b, a', b', a$  is a directed circuit,  $a$  and  $a'$  are joined either way and  $b$  and  $b'$  are joined either way; that is, a *replicate*  $R = R(G, e)$  is a tournament with  $V(R) = V(G) \cup \{a', b'\}$  and

$$\begin{aligned} E(R) = E(G) &\cup \{(a', v) : v \in N_{\text{out}}(a) \setminus \{b\}\} \cup \{(v, a') : v \in N_{\text{in}}(a)\} \\ &\cup \{(b', v) : v \in N_{\text{out}}(b)\} \cup \{(v, b') : v \in N_{\text{in}}(b) \setminus \{a\}\} \\ &\cup \{(b, a'), (a', b'), (b', a)\} \cup \{\text{exactly one of } (a, a'), (a', a)\} \\ &\cup \{\text{exactly one of } (b, b'), (b', b)\}. \end{aligned}$$

We observe that for each arc  $e$ , there are four nonidentical replicates  $R(G, e)$  that we may construct (depending on how we orient the edges  $aa', bb'$ ).

**Definition 4.** Let  $G$  be a tournament, and let  $n \geq 1$  be fixed.

(1) An *n-e.c. tournament problem* is a  $2 \times n$  matrix

$$\begin{pmatrix} x_1 & \dots & x_n \\ i_1 & \dots & i_n \end{pmatrix},$$

where  $\{x_1, \dots, x_n\}$  is an  $n$ -element subset of  $V(G)$ , and for  $1 \leq j \leq n$ ,  $i_j \in \{\uparrow, \downarrow\}$ .

(2) A *solution* to an *n-e.c. tournament problem* is a vertex  $z \in V(G)$  such that  $z \in N_{\text{in}}(x_j)$  if  $i_j = \uparrow$  and  $z \in N_{\text{out}}(x_j)$  if  $i_j = \downarrow$ .

Note that a tournament  $G$  is *n-e.c.* if and only if each *n-e.c. tournament problem* in  $G$  has a solution.

**Theorem 5.** If  $G$  is a 2-e.c. tournament, then for every  $e \in E(G)$ , each replicate  $R = R(G, e)$  is 2-e.c.

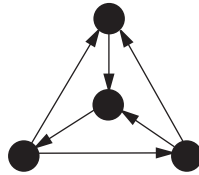


Fig. 3. The unique 1-e.c. tournament of order 4.

**Proof.** Fix  $e = (a, b) \in E(G)$ . Fix distinct  $x, y \in V(R)$ . We show that each problem  $\begin{pmatrix} x & y \\ i & j \end{pmatrix}$ ,  $i, j \in \{\uparrow, \downarrow\}$  has a solution in  $R$ .

Case 1:  $|\{a', b'\} \cap \{x, y\}| = 0$ . A solution to the problem in  $G$  is a solution to the problem in  $R$ .

Case 2:  $|\{a', b'\} \cap \{x, y\}| = 1$ .

Assume that  $x = a'$  and  $y \neq b'$ . First, suppose  $y = a$ . If  $(i, j) = (\uparrow, \uparrow)$ , an in-neighbour of  $a$  in  $G$  solves the problem; if  $(i, j) = (\downarrow, \downarrow)$ , an out-neighbour of  $a$  in  $G$  other than  $b$  solves the problem. The vertex  $b$  solves  $\begin{pmatrix} a' & a \\ \uparrow & \downarrow \end{pmatrix}$  and  $b'$  solves  $\begin{pmatrix} a' & a \\ \downarrow & \uparrow \end{pmatrix}$ .

If  $y \neq a$ , first solve  $\begin{pmatrix} a & y \\ i & j \end{pmatrix}$  by say,  $c$ , in  $G$ . If  $c \neq b$ , then  $c$  also solves  $\begin{pmatrix} a' & y \\ i & j \end{pmatrix}$ . If  $c = b$ , then  $i = \downarrow$  and  $y \neq b$ , so  $b'$  solves  $\begin{pmatrix} a' & y \\ \downarrow & j \end{pmatrix}$ .

The case when  $x = b'$  and  $y \neq a'$  follows by a similar argument.

Case 3:  $|\{a', b'\} \cap \{x, y\}| = 2$ .

Where  $z$  is a solution of  $\begin{pmatrix} a & b \\ i & j \end{pmatrix}$  in  $G$ ,  $z$  is a solution of  $\begin{pmatrix} a' & b' \\ i & j \end{pmatrix}$  in  $R$ .  $\square$

Using tournament replication on  $D_7$ , we obtain 2-e.c. tournaments for any odd order  $k$ ,  $k \geq 7$ . Now we work on finding 2-e.c. tournaments of all possible even orders.

**Theorem 6.** *There is no 2-e.c. tournament of order 8.*

**Proof.** It is straightforward to see that there is a unique 1-e.c. tournament of order 4; see Fig. 3. Let  $G$  be a 2-e.c. tournament of order 8. Then  $G$  has a vertex of degree 4. In fact, the outdegree sequence of  $G$  is completely determined.

**Claim.**  *$G$  has exactly four vertices of indegree 3 and four vertices of indegree 4.*

Let  $v \in V(G)$ . Since both  $G \upharpoonright N_{\text{in}}(v)$  and  $G \upharpoonright N_{\text{out}}(v)$  are 1-e.c., it follows that  $3 \leq |N_{\text{in}}(v)| \leq 4$ . Let  $x$  be the number of vertices of indegree 3, and let  $y$  be the number of vertices of indegree 4. Then since the sum of all indegrees is the number of arcs,

$$\begin{aligned} x + y &= 8, \\ 3x + 4y &= 28. \end{aligned}$$

Solving the system establishes the claim.

Now suppose  $V(G) = \{1, \dots, 8\}$ . For each vertex  $v$  of  $G$ , one of the subgraphs induced by  $N_{\text{in}}(v)$  and  $N_{\text{out}}(v)$  is  $D_3$  and the other is the tournament of Fig. 3.

Without loss of generality, suppose vertex 1 has indegree 4 and the subgraphs induced by  $N_{\text{in}}(1)$  and  $N_{\text{out}}(1)$  are as in Fig. 4.

Case 1: Vertex 8 has indegree 3.

Without loss of generality, by the symmetry of 2, 3, and 4 in the directed graph in Fig. 4,  $(2, 8), (3, 8), (8, 4) \in E(G)$ .  $N_{\text{in}}(8) = \{2, 3, 7\}$  and  $(2, 3) \in E(G)$ , so for  $G \upharpoonright N_{\text{in}}(8) \cong D_3$ , also  $(3, 7), (7, 2) \in E(G)$ .  $N_{\text{out}}(8) = \{1, 4, 5, 6\}$  and  $(5, 1), (5, 6) \in E(G)$ , so  $(4, 5) \in E(G)$ .

Now  $|N_{\text{out}}(7)| = 4$ , so all remaining arcs meeting 7 must be directed toward 7, so  $(4, 7) \in E(G)$ . Then  $N_{\text{in}}(7) = \{3, 4, 6\}$  and  $(3, 4) \in E(G)$ , so  $(4, 6), (6, 3) \in E(G)$ . The vertices 4, 5, and 8 are in  $N_{\text{in}}(6)$ , but  $(8, 5), (4, 5) \in E(G)$  so  $|N_{\text{in}}(6)| = 4$ , so  $(2, 6) \in E(G)$ . Now  $N_{\text{in}}(6) = \{2, 4, 5, 8\}$  and  $(4, 5), (8, 5) \in E(G)$  so  $(5, 2) \in E(G)$ .

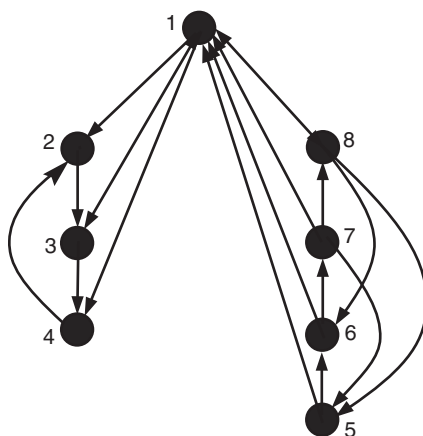
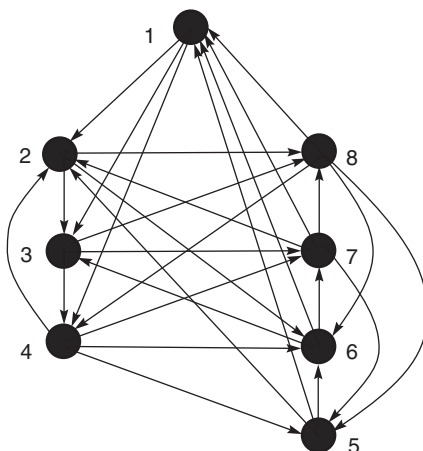


Fig. 4. The in- and out-neighbours of 1.

Fig. 5.  $G$  missing one arc.

Now we have all but one arc of  $G$ , either  $(3, 5)$  or  $(5, 3)$ . See Fig. 5. If that arc were  $(3, 5)$ , then  $N_{\text{out}}(3) = \{4, 5, 7, 8\}$  and  $(4, 5), (7, 5), (8, 5) \in E(G)$  which is a contradiction. Otherwise, if that arc were  $(5, 3)$ , then  $N_{\text{out}}(5) = \{1, 2, 3, 6\}$  and  $(1, 3), (2, 3), (6, 3) \in E(G)$ , which is a contradiction.

Case 2: Vertex 8 has indegree 4.

In this case  $(2, 8), (3, 8), (4, 8) \in E(G)$ . Then  $N_{\text{out}}(8) = \{1, 5, 6\}$ , but  $(5, 1), (6, 1) \in E(G)$ , which is a contradiction.  $\square$

To find 2-e.c. tournaments of all possible even orders as described in Theorem 3, it is sufficient to give an example of a 2-e.c. tournament of order 10, and then use replication. For this, see the tournament  $R'$  in Fig. 6. It is straightforward to verify that  $R'$  is 2-e.c.: one need only check the vertices 1, 2, and 10 versus each of the other vertices. The details are tedious and are therefore omitted.

In [12] it was proved that whenever there is a 2-e.c. graph of order  $m$ , then there is a 2-e.c. graph of order  $m + 1$ , and the question of this type of monotonicity was raised in general for  $n$ -e.c. graphs. We remark that the “gap” for 2-e.c. tournaments supplies the first example of nonmonotonicity of a 2-e.c. property.

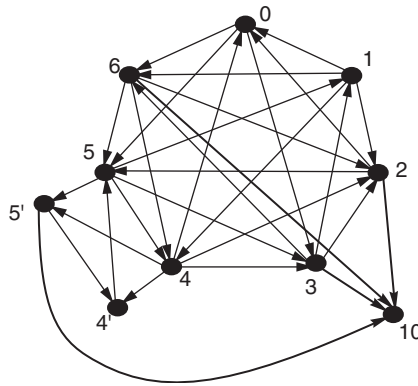


Fig. 6. The tournament  $R'$ . Reverse the arc  $(2, 1)$  in  $R(D_7, (5, 4))$  (where  $(4, 4')$  and  $(5, 5')$  are arcs), and add a new vertex 10 so that  $N_{\text{in}}(10) = \{2, 3, 5', 6\}$ . Note that not all arcs are shown.

#### 4. Examples of 2-e.c. critical tournaments

**Definition 5.** An arc  $e = (a, b)$  of tournament  $G$  is *good* if every vertex  $v \neq a, b$  is the unique solution to some 2-e.c. tournament problem not involving  $a$  or  $b$ .

**Lemma 7.** Let  $G$  be a 2-e.c. critical tournament and let arc  $e = (a, b)$  be good. Then each replicate  $R = R(G, e)$  is a 2-e.c. critical tournament.

**Proof.** Note that: the unique solution of  $\begin{pmatrix} b & b' \\ \uparrow & \downarrow \end{pmatrix}$  is  $a$ , of  $\begin{pmatrix} b & b' \\ \downarrow & \uparrow \end{pmatrix}$  is  $a'$ , of  $\begin{pmatrix} a & a' \\ \uparrow & \downarrow \end{pmatrix}$  is  $b'$ , and of  $\begin{pmatrix} a & a' \\ \downarrow & \uparrow \end{pmatrix}$  is  $b$ . Now let  $x \in V(R) - \{a, a', b, b'\}$ . By hypothesis,  $x$  is the unique solution to some 2-e.c. tournament problem in  $G$ . If  $a'$  were a solution to this problem in  $R$  then  $a$  would be a solution to it in  $G$ , and if  $b'$  were a solution to this problem in  $R$ , then  $b$  would be a solution to it in  $G$ . Therefore,  $x$  is the unique solution to this problem in  $R$ .  $\square$

**Definition 6.** In the definition of replication of the arc  $e = (a, b)$  in tournament  $G$ , we insist that the arc between  $a$  and  $a'$  be  $(a, a')$ , and the arc between  $b$  and  $b'$  be  $(b', b)$ , then we call the replication a *type-1 replication*, and use a subscript 1 to indicate the resulting tournament,  $R_1(G, e)$ .

**Lemma 8.** Let  $G$  be a 2-e.c. critical tournament and let  $e = (a, b) \in E(G)$  be good. Repeatedly replicating  $e$  using type-1 replication gives a 2-e.c. critical tournament.

**Proof.** Define  $G_0 = G$ . For  $k \geq 0$ , define  $G_{k+1} = R_1(G_k, e)$ , and call the replication arc  $e_{k+1} = (a_{k+1}, b_{k+1})$ . Then by Lemma 5,  $G_{k+1}$  is a 2-e.c. tournament of order  $|V(G)| + 2k$ . We need to show that  $G_{k+1}$  is 2-e.c. critical.

We proceed by induction on  $k$ . Assume  $G_k$  is 2-e.c. critical and that for  $1 \leq j \leq k$ , vertex  $a_j$  uniquely solves  $\begin{pmatrix} b_j & b \\ \uparrow & \downarrow \end{pmatrix}$  and vertex  $b_j$  uniquely solves  $\begin{pmatrix} a_j & a \\ \downarrow & \uparrow \end{pmatrix}$ . Consider  $G_{k+1}$ .

Since  $(a, b)$  is good in  $G$ , each vertex  $v \in V(G) \setminus \{a, b\}$  is the unique solution to some 2-e.c. tournament problem in  $G$  not involving  $a$  or  $b$ . In  $G_{k+1}$ , no vertex  $a_j$  or  $b_j$ ,  $1 \leq j \leq k + 1$  can solve this problem, because otherwise,  $a$  or  $b$  would have solved it in  $G$ .

Vertices  $a_{k+1}$  and  $b_{k+1}$  cannot solve the problems that  $a_j$  and  $b_j$  ( $1 \leq j \leq k$ ) uniquely solve in the induction hypothesis: for the  $a_j$  problem,  $(b_j, a_{k+1})$  and  $(b_{k+1}, b)$  are arcs of  $G_{k+1}$ ; for the  $b_j$  problem,  $(b_{k+1}, a_j)$  and  $(a, a_{k+1})$  are arcs of  $G_{k+1}$ .

Vertex  $a_{k+1}$  uniquely solves  $\begin{pmatrix} b_{k+1} & b \\ \uparrow & \downarrow \end{pmatrix}$  since every vertex except  $a_{k+1}$  and  $a$  (and  $b$ ) is directed the same way with respect to  $b$  and  $b_{k+1}$ , and  $a$  is directed toward  $b$ . By a similar argument, vertex  $b_{k+1}$  uniquely solves  $\begin{pmatrix} a_{k+1} & a \\ \downarrow & \uparrow \end{pmatrix}$ , since every vertex except  $b_{k+1}$  and  $b$  (and  $a$ ) is directed the same way with respect to  $a$  and  $a_{k+1}$ , and  $a$  is directed toward  $b$ .

Finally,  $a$  uniquely solves  $\begin{pmatrix} b_{k+1} & b \\ \downarrow & \uparrow \end{pmatrix}$  and  $b$  uniquely solves  $\begin{pmatrix} a_{k+1} & a \\ \uparrow & \downarrow \end{pmatrix}$ .  $\square$

Using Lemmas 7 and 8, we may construct 2-e.c. critical tournament for all the possible orders  $k$ , where  $k \geq 7$  and  $k \neq 8$  as follows. For the odd orders, we note that in  $D_7$ ,  $(4, 3)$  is a good arc, as demonstrated by the following table.

Vertex	0	1	2	5	6
Uniquely solves	$\begin{pmatrix} 2 & 5 \\ \downarrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 0 & 6 \\ \uparrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 0 & 5 \\ \uparrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 0 & 2 \\ \downarrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ \downarrow & \downarrow \end{pmatrix}$

For the even orders, the following tables demonstrate that the tournament  $R'$  (introduced at the end of Section 3) is 2-e.c. critical, and that  $(0, 6)$  is a good arc.

Vertex	0	1	2	3	4
Uniquely solves	$\begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 5 & 4 \\ \downarrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 4 & 3 \\ \downarrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 5' & 4' \\ \downarrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 5 & 5' \\ \downarrow & \uparrow \end{pmatrix}$

Vertex	5	6	4'	5'	10
Uniquely solves	$\begin{pmatrix} 4 & 4' \\ \uparrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ \downarrow & \downarrow \end{pmatrix}$	$\begin{pmatrix} 5' & 3 \\ \downarrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 4 & 4' \\ \downarrow & \uparrow \end{pmatrix}$	$\begin{pmatrix} 2 & 5' \\ \downarrow & \downarrow \end{pmatrix}$

We close with the following problem: find examples of  $n$ -e.c. tournaments, where  $n \geq 3$ , that are not Paley tournaments.

## Acknowledgements

The authors gratefully acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC).

## References

- [1] R. Aharoni, E. Berger, The number of edges in critical strongly connected graphs, *Discrete Math.* 234 (2001) 119–123.
- [2] W. Ananchuen, L. Caccetta, On the adjacency properties of Paley graphs, *Networks* 23 (1993) 227–236.
- [3] W. Ananchuen, L. Caccetta, On tournaments with a prescribed property, *Ars Combin.* 36 (1993) 89–96.
- [4] C. Berge, *Graphs and Hypergraphs*, North-Holland, Amsterdam, 1973.
- [5] C. Berge, Some Common properties for regularizable graphs, edge-critical graphs and B-graphs, *Theory and Practice of Combinatorics*, North-Holland Mathematical Studies, vol. 60, North-Holland, Amsterdam, 1982, pp. 31–44.
- [6] C. Berge, About vertex-critical nonbicolorable hypergraphs, *Austral. J. Combin.* 9 (1994) 211–219.
- [7] C. Berge, Some properties of non-bicolorable hypergraphs and the four-color problem, *First International Colloquium on Graphs and Optimization (GOI)*, 1992 (Grimentz), *Discrete Appl. Math.* 65 (1996) 73–79.
- [8] B. Bollobás, *Random Graphs*, Academic Press, London, 1985.
- [9] A. Bonato, K. Cameron, On an adjacency property of almost all graphs, *Discrete Math.* 231 (2001) 103–119.
- [10] A. Bonato, K. Cameron, On 2-e.c. line-critical graphs, *J. Combin. Math. Combin. Comput.* 38 (2001) 111–121.
- [11] J.A. Bondy, U.S.R. Murty, *Graph Theory with Applications*, North-Holland, New York, 1976.
- [12] L. Caccetta, P. Erdős, K. Vijayan, A property of random graphs, *Ars Combin.* 19 (1985) 287–294.
- [13] P.J. Cameron, The random graph, in: R.L. Graham, J. Nešetřil (Eds.), *Algorithms and Combinatorics*, vol. 14, Springer, New York, 1997, pp. 333–351.
- [14] P.J. Cameron, D. Stark, A prolific construction of strongly regular graphs with the  $n$ -e.c. property, *Electron. J. Combin.* 9 (1) (2002), Research Paper 31.
- [15] P. Erdős, On a problem in graph theory, *Math. Gaz.* 47 (1963) 220–223.
- [16] T. Gallai, Kritische Graphen I, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 8 (1963) 165–192.
- [17] R.L. Graham, J.H. Spencer, A constructive solution to a tournament problem, *Canad. Math. Bull.* 14 (1971) 45–48.
- [18] J.W. Moon, On subtournaments of a tournament, *Canad. Math. Bull.* 9 (1966) 297–301.
- [19] V. Neumann-Lara, Vertex critical 4-dichromatic circulant tournaments, *Discrete Math.* 170 (1997) 289–291.



- [21] J. Plesník, Diametrically critical tournaments, *Časopis Pěst. Mat.* 100 (1975) 361–370.
- [22] J.H. Schmerl, W.T. Trotter, Critically indecomposable partially ordered sets, graphs, tournaments and other binary relational structures, *Discrete Math.* 113 (1993) 191–205.
- [23] E. Szekeres, G. Szekeres, On a problem of Schütte and Erdős, *Math. Gaz.* 49 (1965) 290–293.
- [24] C. Thomassen, Color-critical graphs on a fixed surface, *J. Combin. Theory Ser. B* 70 (1997) 67–100.
- [25] B. Toft, An investigation of colour-critical graphs with complements of low connectivity, *Advances in Graph Theory* (Cambridge Combinatorial Conference, Trinity Colloquium, Cambridge, 1977); *Ann. Discrete Math.* 3 (1978) 279–287.